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Buckling failure analysis of random composite laminates subjected to random loads

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Abstract

The buckling failure probability evaluation of laminated composite plates subjected to different in-plane random loads is investigated. The material properties, fiber angles and layer thickness of the laminates are treated as base-line random variables (BLRV). The statistics of buckling strengths of the laminates are determined by the buckling analysis of the stochastic finite element method. The buckling failure probabilities of the laminates subjected to random loads are obtained using the statistics of buckling strengths, the probability theories and the probability integration in the load space. The feasibility and accuracy of the present approach are validated using the results obtained by the Monte-Carlo method (MCM). Numerical examples are presented to demonstrate the feasibility and application of the developed procedure and to investigate effects of stochastic dependence between buckling strengths corresponding to different failure modes and between random loads on the reliability of the composite laminates.

Keywords: Composite materials; Laminate; Reliability; Buckling failure; Random loads; Stochastic finite element method

1. Introduction

In recent years, laminated composite materials have become important engineering materials for the construction of automobile, mechanical, space and marine structures. The use of laminated composite materials in designing these structures has resulted in a significant increase in payload, weight reduction, speed maneuverability and durability. In pursuing these achievements, the reliability design of laminated composite structures has thus become an important subject of research. For instance, Sun and Yamada (1978) and Cederbaum and Elishakoff (1990) studied the failure probability of composite laminates with random strength parameters subjected to in-plane loads. Cassenti (1984) studied the first-ply failure probability and failure location of laminated composite beams and plates on the basis of the Weibull

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weakest link hypothesis. Kam and Lin (1992), Lin et al. (1993) and Engelstad and Reddy (1993) studied the reliability of linear or nonlinear laminated composite plates with random material properties subjected to transverse loads. Gurvich and Pipes (1995) used a multi-step failure approach to study the failure probability of laminated composite beams subjected to bending. Boyer et al. (1997) used the FORM to study the design of a composite structure for achieving a specified reliability. Lin et al. (1998) presented a procedure for reliability analysis of laminated composite plates with random material constants and uncertain stacking sequences subject to the failure modes of buckling and/or first-ply failure. Many different failure modes have been considered for the analysis of composite laminates in the literature. However, to the author's knowledge, it seems that apart from the reference by Lin et al. (1998), only the single failure mode was considered in previous study.

The anisotropy of fiber reinforced composites is often designed for some special loading conditions to obtain high reliability, or other properties. On the contrary, anisotropy also brings about high sensitivities to loading conditions. This leads to the necessity to take account of the uncertainties of loads and some other design parameters in the reliability analysis. In many engineering applications, the structures may be subjected to different random loads and the corresponding failure modes may be different as well. If the loads are not applied simultaneously, there should be some stochastic correlation between the structural strengths corresponding to different failure modes. The effects of stochastic dependence between structural strengths corresponding to different failure modes and random loads on the reliability analysis of structures may not be negligible. Thus, more efforts are still needed and, in particular, the combined effects of uncertainties in applied loads, material properties, fiber orientations and plate thickness, on the reliability of composite laminates, should be studied thoroughly if highly reliable composite structures are to be designed. Further, the development of an efficient procedure for reliability assessment of laminated composite plates is still a goal to be striven for.

The objective of this study is to investigate the reliability of the random laminated composite plate with the consideration of multiple buckling failure modes corresponding to different random loads. The uncertainties of composite laminates considered here are random material properties and uncertain stacking sequences. The developed procedure for the reliability analysis is the conjunction of the buckling strength statistics, probability theory, and the numerical integration in the load space. The statistics of buckling strengths are determined by the stochastic finite element method. The correlation coefficient of buckling strengths between different random loads is considered in the reliability analysis. Numerical examples are presented to demonstrate the feasibility and applications of the proposed procedure and investigate effects of stochastic dependence between structural strengths corresponding to different failure modes and random loads on reliability analysis. Results obtained from the Monte-Carlo method (MCM) are used to validate the accuracy of the present procedure.

2. Uncertainties in composite laminates and external loading

As is well known, many uncertainties may exist during the process of, for example, measurement and manufacture. A composite laminate is a stack of layers of fiber-reinforced laminae. The fiber-reinforced laminae are made of fibers and matrix that are of two different materials. The way in which the fibers and matrix materials are assembled to make a lamina, as well as the lay-up and curing of laminae, are complicated processes and may involve a lot of uncertain factors. Therefore, the material properties of a composite laminate are random in nature. In the following stochastic buckling analysis (SBA), the elastic moduli (E_1, E_2, ν_{12}, G_{12}, G_{13}, G_{23}) of the material are treated as independent base-line random variables (BLRV), and their statistics are used to predict the mechanical behavior of composite laminates. It is worth noting that as the determination of the degree of dependence among the BLRV is a difficult though not intractable task, the adoption of the independence assumption can greatly simplify the reliability assessment. In fact, the results obtained from the following reliability assessment will show that the independence assumption is acceptable. Further, fiber orientations and thickness of laminae may fluctuate in the vicinity of the prescribed values, depending on the manufacturing process. It is, therefore, necessary and desirable to investigate the effects of the uncertain stacking sequence on the reliability of composite laminates. Herein, the fiber orientation, θ_i, and the thickness, t_i, of each layer are also considered to be random. The uncertainties of the stacking sequence can be expressed in the following forms (Nakagiri et al., 1986):

$$\theta_i = \overline{\theta}_i(1 + \omega_i) \tag{1a}$$

$$t_i = \overline{t}_i(1 + \eta_i) \quad i = 1, 2, \ldots, N, \tag{1b}$$

where ω_i and η_i stand for random variables for θ_i and t_i, respectively; $\overline{\theta}_i$ and $\overline{t}_i$ are the mean values of the variables θ_i and t_i, respectively; N is the number of layers. It is noted that an uncertain layer thickness can cause uncertainty in the Z coordinates of the layer boundary and centroid.

From now on, $\alpha_i(i = 1, 2, \ldots, 2N + 6)$ will be used to denote the BLRV in which $\alpha_i(i = 1, 2, \ldots, N)$ denote the fiber orientations, $\alpha_i(i = N + 1, \ldots, 2N)$ layer thicknesses, and $\alpha_i(i = 2N + 1, \ldots, 2N + 6)$ the material properties E_1, E_2, v_{12}, G_{12}, G_{13} and G_{23}, respectively. The aforementioned uncertainties in mechanical properties and stacking sequence of composite laminae can cause variations in the constitutive matrix of the laminate.

In many engineering applications, the structures may be subjected to different types of external loads, which may not be applied simultaneously. In general, the magnitudes of the external loads are random in nature. The uncertainties in external loads may have effects on the laminate reliability. For simplicity, here only the in-plane external loads are considered for buckling analysis, and the external loads are assumed to be proportional to one loading parameter λ. Thus, in the following reliability analysis, the loading parameters for the different types of external loads may be regarded as the magnitudes of the corresponding external loads and treated as random variables.

3. Stochastic finite element buckling analysis of composite plates

The present stochastic finite element buckling analysis of laminated composite plates comprising random parameters is based on the first-order shear deformation theory and the mean-centered second-order perturbation technique. Spatial variability is not considered in the stochastic finite element formulation. The shear deformable finite element developed by Kam and Chang (1992) is used in the finite element analysis. The element can be applied to the analyses of both thin and thick plates, and it contains five degrees-of-freedom (three displacements and two slopes, i.e., shear rotation) per node. In evaluating the terms of element stiffness matrix, a quadratic element of the serendipity family and the reduced integration are used. Here, only the in-plane edge external loads are considered. The external loading is assumed to be proportional to one random loading parameter λ, and the effect of the prebuckling displacements is assumed to be negligible. Thus, the buckling analysis considered here is the linear buckling analysis, and the buckling loading parameter B may be determined by solving the eigenvalue problem

$$\underline{K}_b \underline{D} = \lambda \underline{K}_g \underline{D}, \tag{2}$$

where $\underline{K}_b$ is the linear stiffness matrix of the structure, $\underline{K}_g$ is the geometric stiffness matrix of the structure corresponding to $\lambda = 1$, and $\underline{D}$ is the eigenvector (buckling mode). The buckling loading parameter B is the minimum eigenvalue of Eq. (2). In this study, B is referred to as the buckling strength of the laminate. The buckling mode $\underline{D}$, stiffness matrices $\underline{K}_b$ and $\underline{K}_g$ and buckling loading parameter B in Eq. (2) are functions of the random variables $\alpha_i(i = 1, 2, \ldots, 2N + 6)$, which represent structural uncertainties in the structure. In this study, $(\cdot)^{(0)}$ denotes the value of $(\cdot)$ calculated at $\overline{\alpha}_k (k = 1, 2, \ldots, m)$, in which $\overline{\alpha}_k$ is the mean of α_k, and $m = 2N + 6$; $(\cdot)^{(1)}_{,i}$ denotes the values of the first-order derivatives of $(\cdot)$ with respect to random variables α_i calculated at $\overline{\alpha}_k$; and $(\cdot)^{(2)}_{,ij}$ denotes the values of second-order derivatives of $(\cdot)$ with respect to random variables α_i and α_j calculated at $\overline{\alpha}_k$. Based on the mean-centered second-order perturbation technique, the second-order approximate mean and first-order approximate variance of the buckling strength may be expressed as:

$$E[B] \cong B^{(0)} + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} B^{(2)}_{,ij} \operatorname{cov}(\alpha_i, \alpha_j), \tag{3}$$

$$\operatorname{var}[B] \cong \sum_{i=1}^{m} \sum_{j=1}^{m} B^{(1)}_{,i} B^{(1)}_{,j} \operatorname{cov}(\alpha_i, \alpha_j). \tag{4}$$

Here, $B^{(0)}$ is obtained from Eq. (2) by the inverse power method; $B^{(1)}_{,i}$ and $B^{(2)}_{,ij}$ are calculated by using the method proposed in Lin and Kam (1992) and Rudisill (1974), and may be expressed as

$$B^{(1)}_{,i} = \frac{\underline{D}^{(0)^t} (\underline{K}^{(1)}_{\mathrm{b},i} - B^{(0)} \underline{K}^{(1)}_{\mathrm{g},i}) \underline{D}^{(0)}}{\underline{D}^{(0)^t} \underline{K}^{(0)}_{\mathrm{g}} \underline{D}^{(0)}}, \tag{5}$$

$$B^{(2)}_{,ij} = \frac{H}{\underline{D}^{(0)^t} \underline{K}^{(0)}_{\mathrm{g}} \underline{D}^{(0)}} \tag{6a}$$

in which

$$\begin{aligned} \mathrm{H} = {} & \underline{D}^{(0)^t} (\underline{K}^{(2)}_{\mathrm{b},ij} - B^{(0)} \underline{K}^{(2)}_{\mathrm{g},ij} - B^{(1)}_{,j} \underline{K}^{(1)}_{\mathrm{g},i} - B^{(1)}_{,i} \underline{K}^{(1)}_{\mathrm{g},j}) \underline{D}^{(0)} + \underline{D}^{(0)^t} (\underline{K}^{(1)}_{\mathrm{g},i} - B^{(0)} \underline{K}^{(1)}_{\mathrm{g},i} - B^{(1)}_{,i} \underline{K}^{(0)}_{\mathrm{g}}) \underline{D}^{(1)}_{,j} \\ & + \underline{D}^{(0)^t} (\underline{K}^{(1)}_{\mathrm{b},j} - B^{(0)} \underline{K}^{(1)}_{\mathrm{g},j} - B^{(1)}_{,j} \underline{K}^{(0)}_{\mathrm{g}}) \underline{D}^{(1)}_{,i}. \end{aligned} \tag{6b}$$

Using the derivative of Eq. (2) and the derivative of the orthonormalized constraint $\underline{D}^t \underline{D} = 1$ with respect to α_i, the value of $\underline{D}^{(1)}_{,i}$ in Eq. (6b) may be calculated by

$$\underline{\mathrm{C}} \underline{\mathrm{D}}^{(1)}_{,i} = -\underline{\mathrm{E}} \underline{\mathrm{D}}^{(0)}, \tag{7a}$$

where

$$\underline{\mathbf{C}} = \begin{bmatrix} \underline{\mathbf{K}}^{(\mathbf{0})}_{\mathbf{b}} - \lambda^{(0)} \underline{\mathbf{K}}^{(\mathbf{0})}_{\mathbf{g}} & \underline{\mathbf{D}}^{(0)} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{K}}^{(\mathbf{0})}_{\mathbf{b}} - \lambda^{(0)} \underline{\mathbf{K}}^{(\mathbf{0})}_{\mathbf{g}} \\ \underline{\mathbf{D}}^{(0)^t} \end{bmatrix}, \tag{7b}$$

$$\underline{E} = \begin{bmatrix} \underline{K}^{(0)}_{\mathrm{b}} - \lambda^{(0)} \underline{K}^{(0)}_{\mathrm{g}} & \underline{D}^{(0)} \end{bmatrix} \begin{bmatrix} \underline{K}^{(1)}_{\mathrm{b},i} - \lambda^{(1)}_{,i} \underline{K}^{(0)}_{\mathrm{g}} - \lambda^{(0)} \underline{K}^{(0)}_{\mathrm{g},i} \\ \underline{0} \end{bmatrix} \tag{7c}$$

in which $\underline{C}$ and $\underline{E}$ are $n \times n$ matrices, and $\underline{0}$ is a zero matrix of order $1 \times n$, in which n is the order of the square matrices $\underline{K}_{\mathrm{b}}$ and $\underline{K}_{\mathrm{g}}$.

When the BLRV are stochastically independent, $\operatorname{cov}(\alpha_i, \alpha_j) = 0,\ \ i \neq j$, and $\operatorname{cov}[\alpha_i, \alpha_j] = \operatorname{var}[\alpha_i],\ \ i = j$. Thus Eqs. (3) and (4) may be rewritten as

$$E[B] \cong B^{(0)} + \frac{1}{2} \sum_{i=1}^{m} B^{(2)}_{,ii} \operatorname{var}[\alpha_i], \tag{8}$$

$$\operatorname{var}[B] \cong \sum_{i=1}^{m} (B^{(1)}_{,i})^2 \operatorname{var}[\alpha_i]. \tag{9}$$

4. Reliability analysis

Consider the laminated plate subjected to different in-plane loads referred to as load **I** and load **II**, which are not applied simultaneously. Let λ_{I} and λ_{II} denote the loading parameters corresponding to load **I** and load **II**, respectively. In view of Eq. (2), B_{I} and B_{II} are functions of the same base-line system parameters. Thus, the stochastic dependence between B_{I} and B_{II} should be considered in the reliability analysis. The correlation between B_{I} and B_{II} may be estimated from the covariance of B_{I} and B_{II}:

$$\operatorname{cov}[B_{\mathrm{I}}, B_{\mathrm{II}}] \cong \sum_{i=1}^{m}\sum_{j=1}^{m} B_{\mathrm{I},i}^{(1)} B_{\mathrm{II},j}^{(1)} \operatorname{cov}[\alpha_i, \alpha_j], \tag{10}$$

where $\operatorname{cov}[\cdot]$ denotes covariance, $B_{\mathrm{I},i}^{(1)}$ and $B_{\mathrm{II},j}^{(1)}$ can be determined in the previous section. The coefficient of correlation for B_{I} and B_{II} is thus obtained as

$$\rho_{\underline{B}} = \frac{\operatorname{cov}[B_{\mathrm{I}}, B_{\mathrm{II}}]}{\Delta_{B_{\mathrm{I}}}\Delta_{B_{\mathrm{II}}}}, \tag{11}$$

where Δ denotes standard deviation.

The reliability assessment of a composite structure, in general, requires information on the probability distribution and not just statistical moments of the buckling strength of the structure. In the previous section, however, only statistical moments of buckling strength could be determined, while the types of probability distributions of the buckling strengths are indeterminate. Without loss of generality, here both the probability density functions of B_{I} and B_{II} are assumed to be lognormal distribution and denoted by $f_{B_{\mathrm{I}}}(u)$ and $f_{B_{\mathrm{II}}}(v)$.

In the linear buckling analysis, the limiting state of buckling failure is attained when the buckling strengths are less than the magnitude of the corresponding load. Let ξ and η denote some values of loading parameters λ_{I} and λ_{II} corresponding to loads **I** and **II**. The failure probabilities of the laminate subjected to buckling failure corresponding to ξ and η, respectively, are defined as

$$P_d[C_1] = \mathrm{Prob}\,[\text{laminate fails due to load } \mathbf{I}] = \int_0^{\xi} f_{B_{\mathrm{I}}}(u)\,\mathrm{d}u, \tag{12}$$

$$P_d[C_2] = \mathrm{Prob}\,[\text{laminate fails due to load } \mathbf{II}] = \int_0^{\eta} f_{B_{\mathrm{II}}}(u)\,\mathrm{d}u. \tag{13}$$

The reliability of the laminate subjected to two buckling failures corresponding to ξ and η is thus expressed as

$$P_{dS} = 1 - P_{df} = 1 - \{P_d[C_1] + P_d[C_2] - P_d[C_1 \cap C_2]\}, \tag{14}$$

where $P_{dS} = P_{dS}(\xi, \eta)$ and $P_{df} = P_{df}(\xi, \eta)$ are the reliability and failure probability of the laminate corresponding to ξ and η. The joint probability in Eq. (14) is expressed as

$$P_d[C_1 \cap C_2] = \frac{1}{2\pi\sqrt{1-\rho_{\underline{B}}^2}} \int_{-\infty}^{x}\int_{-\infty}^{y} \exp\left[-\frac{u^2 - 2\rho_{\underline{B}}\,u\,v + v^2}{2(1-\rho_{\underline{B}}^2)}\right]\mathrm{d}u\,\mathrm{d}v \tag{15a}$$

with

$$x = \frac{\ln\xi - E[\ln B_{\mathrm{I}}]}{\Delta_{\ln B_{\mathrm{I}}}}, \quad y = \frac{\ln\eta - E[\ln B_{\mathrm{II}}]}{\Delta_{\ln B_{\mathrm{II}}}}. \tag{15b}$$

The probability given in Eq. (15) can be calculated using the series expansion (Abramowitz and Stegun, 1967) and given by

$$P[C_1 \cap C_2] = Q(x)Q(y) + \sum_{n=0}^{\infty} \frac{Z^{(n)}(x)Z^{(n)}(y)}{(n+1)!} \rho_{\underline{B}}^{n+1} \tag{16a}$$

with

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\,\mathrm{d}t, \tag{16b}$$

$$Z(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}. \tag{16c}$$

From Eqs. (12) and (13), the failure probabilities of the laminate subjected to either random load **I** and load **II**, respectively, can be expressed by

$$P_r[C_1] = \int_0^{\infty} f_{\lambda_\mathrm{I}}(\xi) P_d[C_1]\,\mathrm{d}\xi, \tag{17}$$

$$P_r[C_2] = \int_0^{\infty} f_{\lambda_\mathrm{II}}(\eta) P_d[C_2]\,\mathrm{d}\eta, \tag{18}$$

where $f_{\lambda_\mathrm{I}}(\xi)$ and $f_{\lambda_\mathrm{II}}(\eta)$ are the probability density functions of the random loading parameters λ_I and λ_II.

The failure probability of the laminate subject to two buckling failures corresponding to random loads **I** and **II** can be expressed by

$$P_{rf} = \int_0^{\infty} \int_0^{\infty} f_{\lambda_\mathrm{I}\lambda_\mathrm{II}}(\xi,\eta) P_{df}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{19}$$

where $P_{df}(\xi,\eta)$ is given in Eq. (14); $f_{\lambda_\mathrm{I}\lambda_\mathrm{II}}(\xi,\eta)$ is the joint probability density of random loading parameters λ_I and λ_II. In this study, both λ_I and λ_II are assumed to be normal variates and the corresponding joint probability density function can be written as

$$\begin{aligned} f_{\lambda_\mathrm{I}\lambda_\mathrm{II}}(\xi,\eta) = {} & \frac{1}{2\pi \Delta_{\lambda_\mathrm{I}} \Delta_{\lambda_\mathrm{II}} \sqrt{1-\rho_{\underline{\lambda}}^2}} \exp\left[\frac{-1}{2(1-\rho_{\underline{\lambda}}^2)} \left\{ \left(\frac{\xi - \overline{\lambda}_\mathrm{I}}{\Delta_{\lambda_\mathrm{I}}} \right)^2 \right.\right. \\ & \left.\left. - 2\rho_{\underline{\lambda}} \left(\frac{\xi - \overline{\lambda}_\mathrm{I}}{\Delta_{\lambda_\mathrm{I}}} \right) \left(\frac{\eta - \overline{\lambda}_\mathrm{II}}{\Delta_{\lambda_\mathrm{II}}} \right) + \left(\frac{\eta - \overline{\lambda}_\mathrm{II}}{\Delta_{\lambda_\mathrm{II}}} \right)^2 \right\} \right], \end{aligned} \tag{20}$$

where $\overline{\lambda}_i$ and $\Delta_{\lambda i}$ (i=**I**, **II**) denote the mean values and standard deviations of the random loading parameters λ_i corresponding to load i; $\rho_{\underline{\lambda}}$ denotes the correlation coefficient for λ_I and λ_II.

The probability of Eqs. (17)–(19) can be evaluated by the numerical integration. The Monte-Carlo simulation using the random multivariate numbers generated by the routine RNMVN of IMSL (1989) mathematical package will be employed to verify the accuracy of the present reliability assessment of the random laminated plate subjected to random loads.

5. Experimental investigation

The statistics of material properties and lamina thickness required for the present analytical method were obtained from experimental measurements and testing. The material used in the present study was graphite/epoxy (Q-1115) prepreg tapes supplied by the Toho Co., Japan. A number of 30×30 cm^2 composite laminates were made using the vacuum bag molding method in which the vacuum bagged laminate was cured by a hot press machine. Each cured laminate was then cut to make different types of specimens used for material characterization. The material properties were determined from experiments conducted in accordance with the relevant ASTM standards (1990). The statistics of each lamina material parameter was determined from a set of 17 specimens which were prepared from 17 different laminates. Thicknesses of different laminated plates were measured and the results used to determine the statistics of plate thickness as well as those of lamina thickness. The experimentally determined statistics of material properties and lamina thickness as well as the confidence intervals for their means with 95% confidence level are tabulated in Table 1. Spatial variabilities of the BLRV within a laminated plate were also studied experimentally. Thicknesses at different locations on a laminate were measured and same specimens cut from a laminate were used to determine each of the lamina parameters. The experimental results showed that the spatial variabilities of the BLRV within a laminate were insignificant (coefficient of variation (COV) less than 1%).

A number of $[0°/90°/0°/90°]_{2S}$ square laminates of size 10×10 cm^2 were subjected to axial buckling tests using a 10-ton Instron testing machine. The top and bottom edges of the laminates were clamped during the test. Herein, around 20 specimens were tested. The test results were fitted by normal, lognormal or Weibull distribution distributions as shown in Fig. 1. The statistics of the buckling loads, the confidence intervals for the mean buckling loads with 95% confidence level, and the chi-squared test statistics determined from the goodness-of-fit tests of different expected distributions of the buckling loads for the laminated composite plates are listed in Table 2. Both the confidence intervals for the means and the variations (COV less than 6%) of the buckling loads of the laminated plates are small. When comparing the chi-squared test statistics listed in Table 2, it is obvious that the normal and lognormal models are the better representation

Table 1
Statistics of base-line random variables

Random variables	95% Confidence interval for mean	COV (%)
E_1	138.41 ± 3.38 Gpa	3.60
E_2	9.24 ± 0.27 GPa	4.34
G_{12}	4.52 ± 0.17 GPa	5.51
G_{13}	4.52 ± 0.17 GPa	5.51
G_{23}	1.02 ± 0.04 GPa	5.51
v_{12}	0.32 ± 0.01	5.90
t_i	0.120 ± 0.001 mm	1.49

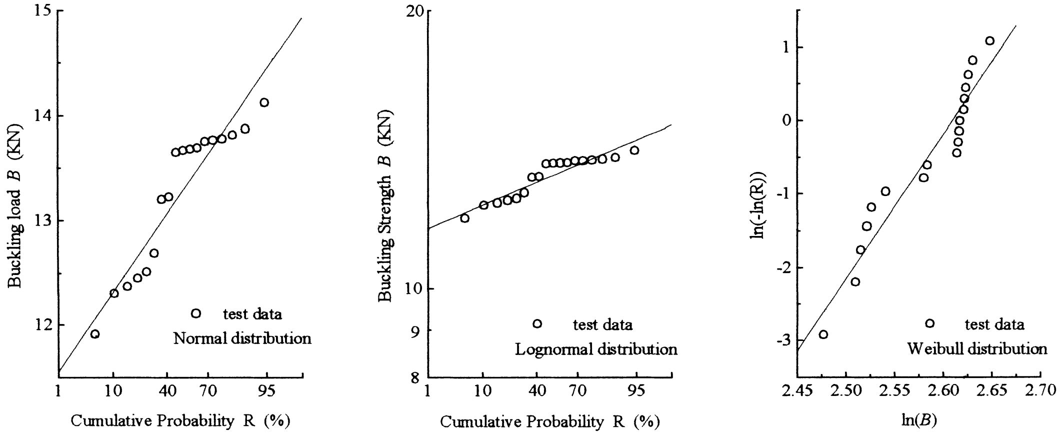

Fig. 1. The cumulative probability of the buckling test data fitted by normal, lognormal, and Weibull distribution.

Table 2
Experimental statistics of buckling load for composite plates [0°/90°/0°/90°]$_{2S}$

Statistics		Chi-squared test statistics[a]		
Mean (kN)	COV (%)	Normal	Lognormal	Weibull
13.24 ± 0.36[b]	5.57	15.5	14.9	19.7

[a] For $k = 6$, $m = 2$, and the α risk at 0.005, $\chi^2(1 - \alpha; k - m - 1) = 12.8$ (Neter et al., 1988).
[b] 95% Confidence intervals for mean.

of the sample data for the plates, because they yield the smaller value of the test statistic. The parameters of the experimental distributions of the buckling loads of the laminated composite plates were determined from the sample data using the maximum likelihood method (Neter et al., 1988).

6. Numerical studies

In this article, the standard deviation of the fiber angles is assumed to be three degree. The BLRV considered here are: material properties, layer thickness and fiber angles of layers. Unless otherwise stated, the statistics of BLRV given in Table 1 are used for numerical studies.

To verify the accuracy of the present SBA, the first example considered is the laminated plate studied experimentally in the previous section. The boundary conditions and the applied load are shown in Fig. 2(a). A 4×4 element mesh is used for discretization. The theoretical statistics of the buckling load corresponding to different BLRV are given in Table 3. The mean values given in Table 3 are the second order approximations of the mean value of buckling load. The corresponding first order approximation of the mean value of buckling load is 14.21 kN. It is noted that the discrepancy between the mean value of buckling loads corresponding to the first and second approximation is small (<0.9%). It can be seen that the effect of the random fiber angles on the variation of buckling load is insignificant for this example. When comparing the results given in Table 3 with those given in Table 2, one can see that the agreement between the statistics of buckling load obtained from the present numerical study and experimental study is very good.

The second example considered here is a size 10×10 cm^2 of symmetric angle-ply laminate $[\theta/-\theta]_S$ shown in Fig. 2(b). The laminate is subjected to two in-plane loads: (1) a uniform compressive line load

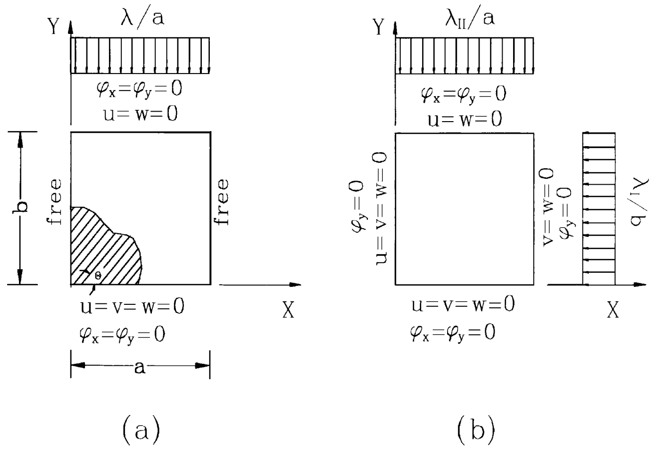

Fig. 2. The geometry, boundary and loading conditions of the laminates.

Table 3
Theoretical statistics of buckling load for the [0°/90°/0°/90°]$_{2S}$ composite plate with various types of base-line random variables

Random variables	Mean (kN)	COV (%)
Materials	14.199	2.92
Lamina thicknesses	14.298	4.33
Fiber angles	14.072	<0.001
Combination	14.150 (6.87%)[a]	5.43 (6.3%)[a]

[a] Percentage error = ((SFEM − experiment)/experiment) × 100.

$0.1\lambda_{\mathrm{I}}$ N/cm in the X-direction, referred to as load **I**, (2) a uniform compressive line load $0.1\lambda_{\mathrm{II}}$ N/cm in the Y-direction, referred to as load **II**. *Note again that load* **I** *and* **II** *are not applied simultaneously here.* Here λ_{I} and λ_{II} are assumed to be normal variates.

First, the effects of different BLRV on the statistics of buckling strength of the laminate are investigated. Figs. 3–5 show the statistics of the laminate buckling strength, which are obtained by using the present SBA and MCM. It is noted that the results obtained by the present SBA are in excellent agreement with those obtained by the MCM in which over 1000 data have been generated for each case. Fig. 3 shows the COV

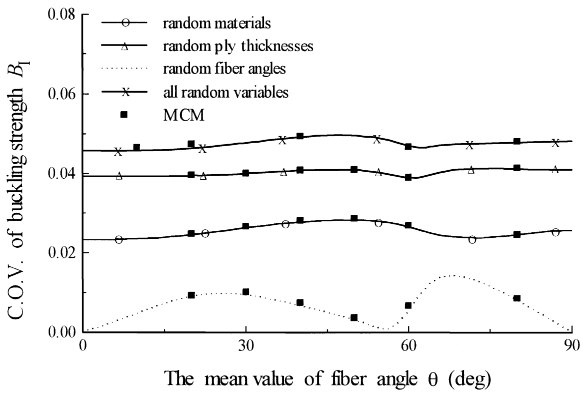

Fig. 3. Effect of different BLRV on the variations of buckling strengths for the laminates $[\theta/-\theta]_{\mathrm{S}}$.

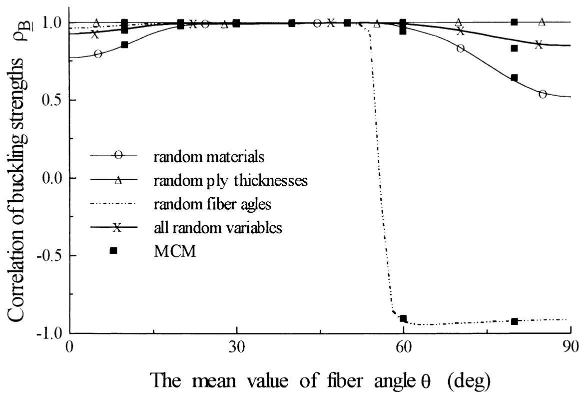

Fig. 4. The correlation coefficients of buckling strengths versus the fiber angle of the laminate $[\theta/-\theta]_{\mathrm{S}}$ with different base-line random parameters.

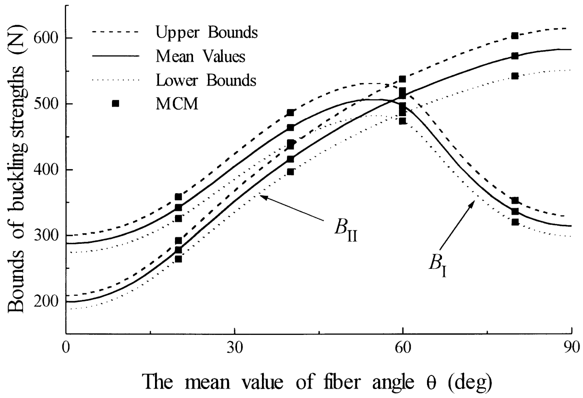

Fig. 5. The mean values and one-standard-deviation bounds of buckling strengths for the laminate subjected to load **I** or load **II**.

for the buckling strength B_I of the laminate with different BLRV. As can seen from Fig. 3, among different BLRV, the randomness of layer thickness has greater effects on the variation of the laminate buckling strength. Fig. 4 shows the correlation coefficient of $\rho_{\underline{B}}$ between buckling strengths B_I and B_{II}. It can be seen that the stochastic dependence between B_I and B_{II} corresponding to random ply thickness is strong positive for all fiber angles. The stochastic dependence between B_I and B_{II} corresponding to random fiber angle is positive for fiber angle $\theta \leqslant 50°$, and is negative for fiber angle $50° < \theta < 90°$. Fig. 5 shows the mean values and the one-standard-deviation bounds of the buckling strength of the laminate with all BLRV corresponding load **I** and load **II**. It can be seen that the values of B_{II} monotonically increase with the increase of the fiber angle, and the value of B_I increases when $\theta \leqslant 60°$ and then decreases with the increase of fiber angle θ.

Next, the reliability of the laminate corresponding to one and two buckling failure modes are studied. Tables 4 and 5 show the reliability of the laminate with all BLRV subjected to load **I** and load **II**. It should be noted that the results given in Table 4 are corresponding to values of loading parameters λ_I and λ_{II} given in it. In other words, load **I** and load **II** are regarded as deterministic loads for this case. The definitions of P_{df}, $P_d[C_1]$ and $P_d[C_2]$ are given in Eqs. (12)–(14). Table 5 shows the reliability of the laminates subjected to stochastic independent random loads with different mean values of loading parameters, $\bar{\lambda}_I$ and $\bar{\lambda}_{II}$. The COV of loading parameters, λ_I and λ_{II} is assumed to be 10%. It can be seen that the reliability of the laminate corresponding to two buckling failure modes is lower than those corresponding to one buckling failure mode. The percentage difference between the results of the present method and MCM are given in

Table 4
The reliabilities of the laminates subjected to given values of loading parameters

Laminates	$(\lambda_I, \lambda_{II})$	Method	$1 - P_{df}$	$1 - P_d[C_1]$	$1 - P_d[C_2]$
$[20°/-20°]_S$	(300 N, 250 N)	Present	0.983 (0.4%)[a]	0.993 (0.4%)	0.983 (0.4%)
		MCM	0.979	0.997	0.979
$[40°/-40°]_S$	(420 N, 380 N)	Present	0.962 (0.6%)	0.968 (0.6%)	0.962 (0.6%)
		MCM	0.968	0.974	0.968
$[60°/-60°]_S$	(450 N, 450 N)	Present	0.976 (0.4%)	0.976 (0.4%)	0.990 (0.3%)
		MCM	0.981	0.981	0.993
$[80°/-80°]_S$	(300 N, 500 N)	Present	0.982 (0.5%)	0.983 (0.5%)	0.990 (0.4%)
		MCM	0.986	0.988	0.994

[a] Percentage error = ((present − MCM)/present) × 100.

Table 5
The reliability of the laminates subjected to stochastically independent random loads

Laminates	$(\bar{\lambda}_{\mathrm{I}}, \bar{\lambda}_{\mathrm{II}})$	Method	$1-P_{rf}$	$1-P_r[C_1]$	$1-P_r[C_2]$
$[20°/-20°]_S$	(300 N, 250 N)	Present	0.767 (0.8%)[a]	0.897 (1.3%)	0.845 (1.8%)
		MCM	0.773	0.909	0.838
$[40°/-40°]_S$	(420 N, 380 N)	Present	0.683 (0.1%)	0.828 (0.6%)	0.804 (0.1%)
		MCM	0.684	0.833	0.803
$[60°/-60°]_S$	(450 N, 450 N)	Present	0.749 (0.1%)	0.835 (0.1%)	0.882 (0.1%)
		MCM	0.748	0.836	0.880
$[80°/-80°]_S$	(300 N, 500 N)	Present	0.772 (0.5%)	0.856 (0.7%)	0.890 (0.3%)
		MCM	0.776	0.862	0.887

[a] Percentage error = ((present − MCM)/present) × 100.

parentheses in Tables 4 and 5. It is noted that the results obtained by the present method are in excellent agreement with those obtained by the MCM in which over 5000 data have been generated for each case.

Next, let us consider the effects of the variations of the stochastic independent random loads and BLRV on the reliability of the laminates. The mean values of λ_{I} and λ_{II} used here are all 450 N. Fig. 6 shows the reliability of a $[60°/-60°]_S$ laminate with different BLRV versus COV of two stochastic independent random loads. It can be seen that when the COV of λ_{I} and λ_{II} is greater than 20%, the discrepancy among the reliabilities of the laminate corresponding to different BLRV are negligible. It seems that when the variation of the random loads is higher than a certain value, the reliability of the laminate may be dominated by the values of the COV of random loads, and may be irrelevant to the type of BLRV. Fig. 7 shows the reliability of a $[60°/-60°]_S$ laminate with different COV of all BLRV versus COV of two stochastic independent random loads. The mean values of BLRV used here are given in Table 1, and the COV of all BLRV considered here are 0%, 5%, 10%, and 15%, respectively. It can be seen that the reliability of the laminate decreases with the increase of the COV of BLRV and loading parameters when the COV of λ_{I} and λ_{II} is less than 20%. It is noted that the reliability level of the laminate is less than about 0.5, and the reliability of the laminate corresponding to COV of BLRV = 15% is higher than that corresponding to COV of BLRV = 10%, when the COV of λ_{I} and λ_{II} is greater than 20%. However, the discrepancy between the reliability corresponding to different values of COV of BLRV is small, when COV of λ_{I} and λ_{II} is greater than 20%. This observation is consistent with that for Fig. 6. From the results presented in Figs. 6 and 7,

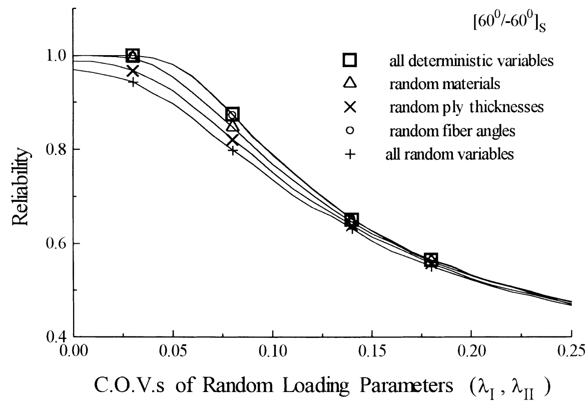

Fig. 6. The reliability versus variations of the stochastic independent random loading parameters for the $[60°/-60°]_S$ laminate composed of different BLRV.

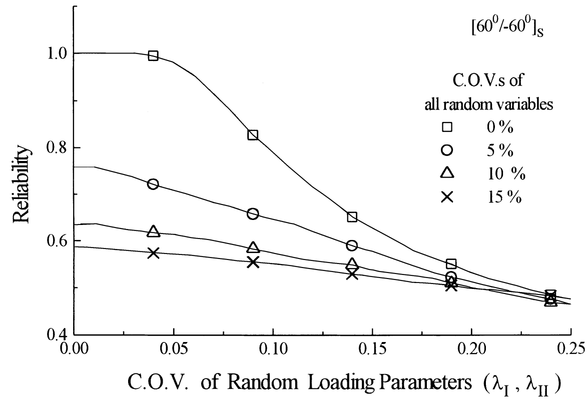

Fig. 7. The reliability versus variations of the stochastic independent random loading parameters for the $[60°/{-60°}]_S$ laminate composed of all BLRV.

one may conclude that when the COV of the stochastically independent random loads and BLRV are all small, the reliabilities of the laminates are dominated by the values of the COV of BLVR. Therefore, all random variables including those of BLRV and random loads have to be considered into the corresponding reliability assessment if highly reliable composite structures are to be designed. But when the variation of the stochastic independent random loads is higher than a certain level (around 20% studied in Figs. 6 and 7), the reliability of the laminate is dominated by the values of the COV of random loading parameters, and may be irrelevant to the types of BLRV and values of the COV of BLRV. According to the above studies, when the variation of the stochastic independent random loads is higher than a certain level, the corresponding reliability can be estimated with a reasonable accuracy by considering only the effects of random loads.

Finally, the effects of the correlation coefficients of the stochastically dependent random loads on the reliability of the laminates are studied. The mean values and COV of λ_I and λ_{II} used here are all 450 N and 10%, respectively. Fig. 8 shows the reliability of a $[60°/{-60°}]_S$ laminate with different BLRV versus correlation coefficients of λ_I and λ_{II}. Fig. 9 shows the reliability of the laminate with all BLRV versus the fiber

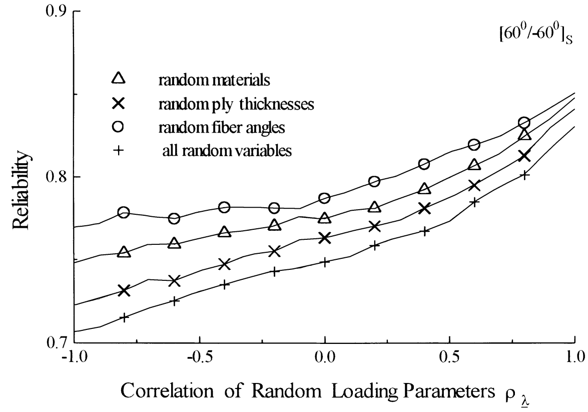

Fig. 8. The reliability versus correlation coefficients of random loading parameters for the $[60°/{-60°}]_S$ laminate composed of different base-line random parameters.

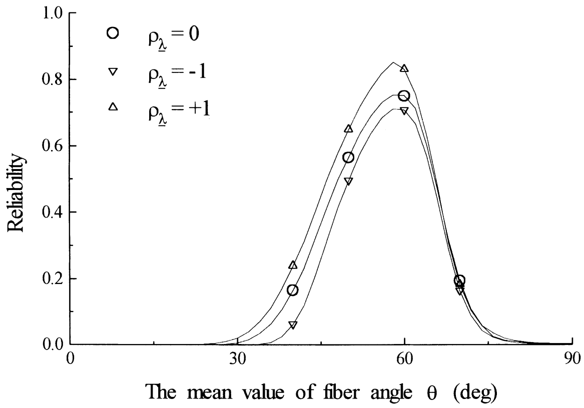

Fig. 9. The reliabilities versus the fiber angle of the laminate $[\theta / -\theta]_S$ with all base-line random parameters.

angles. As can be seen from Figs. 8 and 9, the reliability of the laminate corresponding to positively correlated random loads are higher than those of negatively correlated random loads.

7. Conclusions

The reliability of composite laminates with single or multiple buckling failure modes has been investigated on the basis of the buckling analysis of stochastic finite element method. The accuracy of the SBA in predicting statistics of buckling strengths has been verified by the experimental results and the MCM. The applications of the proposed procedure have been demonstrated by the reliability predictions of symmetric angle-ply laminates with different types of buckling failure modes corresponding to different in-plane edge random loads which are not applied simultaneously. It has been shown that the variations of ply thickness have the greatest effects on the variations of the laminate buckling strengths as well as laminate reliability. Thus, tight control on ply thickness variations may be essential for achieving high reliability. Moreover, the effect of the correlation between random loads has also been investigated and the reliability of the laminates increases with the increase of the correlation coefficients of random loads. From the accuracy of the present results, it is believed that the developed procedure may be valuable for the reliability analysis of composite laminates.

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